State of the Art and Perspectives

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PLAN

- Legal Analysis
  - Law & Economics
  - Computational Law
- Abstract Argumentation
- Sequential Decision Processes
- Perspectives

Definition

Studying law and its consequencies with economic tools.

Law & Economics

- First apparition with Hume and Smith [Smi59, Hum39]
- Rise and seminal works in 1960’ with Posner
- Two schools: Economic Analysis of Law vs Institutionalism

Economic Analysis of Law

- Normative: Law MUST be an incitation mechanism for economic purposes. [PP11]
- Some decisions were *optimal* w.r.t. economic principles [Coa60]
- “the principal function of accident law is to reduce the sum of the cost of accident and the cost of avoiding accidents” [CAL70]

Institutionalism

- Incitative: Law is an incitation and a way to solve conflicts [ST03]
- Behavior school of economy [Sti87, JST98, Sim66, KT79, TK74]
- Two ways studies: legal aspect into economic [DL09]

What does the legal experts think about it ?

- Large adoption in US (due to Common Law)
- Europe is reluctant about the Economic Analysis of Law [DM06, Cap06]
- Some legal experts admit studies should be done but lack of qualification and time [Can05]
- Law = Finance 1970 => should evolve toward risk management using finance-like tools [KBJ14]

Hermeneutic revolution

Is judging a rational and objective action ?

- Hot topic among jurists since mid 20th century: “Legalist vs Attitudism (realism)”
- The trend is definitely a big “No”:
    - Selection among the best alternatives [Fry05]
    - Iterative process because the law is too general [Tro01]
    - There exists disruption in the law due e.g. to new technologies [Sun98]
    - Impact of judges preferences
    - Cognitive bias

Note that the Economic Analysis of Law is legalist.

TAXONOMY

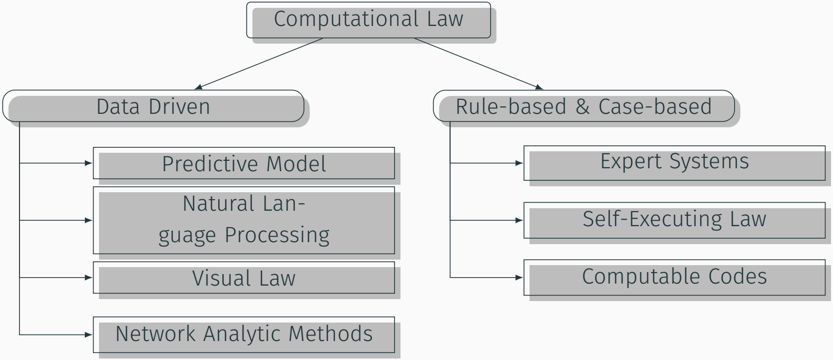


Figure 1: Taxonomy of Computational Law

Ideal Court Assumption

The judges are perfectly rational, omniscient, free of bias.

$\implies$ the decisions are not correlated
$\implies$ impossible to use past case information
$\implies$ expert rules-base systems is the only solution

Predictive Models

Detecting patterns into decision sequences. Blind to legal doctrine. Unable to make justification.

Large focus on SCOTUS...

Reference: 75.4% prediction by legal experts [TWRQ04].

- The Block Model [GSP11]: social network and affiliation network techniques. 77% prediction, not fully predictive, no explanation. Shown a decrease in predictibility in time.
- The Decision Tree Model [ADMR04, TWRQ04]: 6 case features, better predictibility than experts. The experts: better on the vote of the most extremely ideologically oriented judges.

PREDICTIVE MODELS

Reference: 75.4% prediction by legal experts [TWRQ04].

- The Extra-Tree Model [KBJ14]: 67% predictibility but... over 60 years, global and stable model.
  - Court information, Case information, Non-legal factors
  - Weights: a step for an explanation.
  - Predictive power: 23% case, 5% court, +70% non-legal factors $\implies$ argument for realism
- NLP [ATPPL16]: 79% predictibility on ECtUR
  - Hypothesis: the case holds textual information to influence the judgement.
  - Bag-of-Word + topic model using LDA
  - Binary classificator (SVM) trained on labelled cases.

Arguments for realism: a lot of non-legal factors but... "Selection effect" [Kle12]

Indicators to capture ideology.

Ideal Point

One or two dimensional point to sumerize the ideology [LC14]. Often "Conservative vs Liberal".

Focus on the "Swing Justice": Median Voter Theorem.

Segal-Cover [SC89, SECS95]

Non-automatic NLP: editor's assessments.
Independent of vote sources: prevent circular reasoning.

Predictibility

Linear regression $\hat{Y} = aX + b$ on several domain.

- Civil Liberties: $r = 0.69$
- Economic: $r = 0.59$

Variation in predictibility depending on Court composition.

Martin-Quinn Score [QM02, QPM06]

Spatial Voting Model + Resolution by MCMC.
Hypothesis: The ideal point evolves in time.

$$z_{t,k,j} = -|\theta_{t,j} - x_k^r|^2 + \varepsilon_{t,k,j}^r + |\theta_{t,j} - x_k^a|^2 - \varepsilon_{t,k,j}^a$$

with

- $\theta_{t,j}$ the ideal point of the judge j at time t.
- x_k^r, x_k^a the location of the revert and support policy.
- $\varepsilon_{t,k,j}^r, \varepsilon_{t,k,j}^a$ a gaussian noise, centered and with a fixed variance.

Isomorphic to an Item Response Model:

$$z_{t,k,j} = \alpha_k + \beta_k \theta_{t,j} + \varepsilon_{t,k,j}$$

Posteriori Estimation:

$$p(\alpha, \beta, \theta \mid V) \propto p(V \mid \alpha, \beta, \theta) p(\alpha, \beta, \theta)$$

- Gaussian for the priors (standard approach)
- Ideal point however as a random walk:

$$\theta_{t,j} \sim \mathcal{N}(\theta_{t-1}, \Delta_{\theta_{t},j}), \ \forall t \in \{\underline{T}_j, ..., \overline{T}_j\}$$

Metropolis-Hasting to sample Z

Results:

- Confirm previous approach.
- 0.8 correlation in average.
- Shows the evolution in time of the preferences.

Still the most widely used measure of ideology.

Extension to the Amicus Effect [SRS14]

Mix between Martin-Quinn method and NLP approach.
Measure the effect of Amicus using Random Utility Model.

$$p(v_{i,j}|\theta_i, \phi_i, \Delta_i, \alpha_i, \beta_i, \gamma_i) = \sigma(\alpha_i + \theta_i^T(\beta_i \phi_i + \gamma_i^a \Delta_i^a + \gamma_i^r \Delta_i^r))$$

with

- ϕ_i the legal arguments in merits briefs
- θ_i justice ideal point
- Δ_i^a, Δ_i^r the mean issue proportions of the amicus briefs supporting or not the revertal
- $\sigma(x) = \frac{e^x}{1+e^x}$, the logistic function

Amici = one agent with utility function
$u((v_{k,j})_{j\in J_k}) = \sum_{j\in J} \mathbb{1}_{(v_{k,j}=s)}(j)$

$$\max_{\Delta} \mathbb{E}_{\Delta}[u((v_{k,j})_{j\in J_k})] - \frac{\epsilon}{2}||\Delta - \theta||_2^2$$

$$= \max_{\Delta} \sum_{j\in J} \sigma(\alpha + \theta_j^T(\beta\phi + \gamma^s\Delta)) - \frac{\epsilon}{2}||\Delta - \theta||_2^2$$

Prior on Δ: $p_{\text{util}}(\Delta) \propto \mathbb{E}_{\Delta}[u((v_{k,j})_{j\in J_k})] + \epsilon(1 - \frac{1}{2}||\Delta - \theta||_2^2)$
(Random Utility Model [**?**])

Final quantity to estimate the parameters:

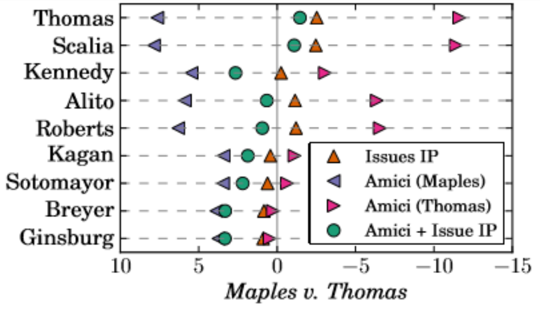
$$\mathcal{L}(w, v, \theta_i, \phi_i, \Delta_i, \alpha_i, \beta_i, \gamma_i)[\prod_{k\in\mathcal{A}} p_{\text{util}}(\Delta_k)]^{\eta}$$

Hard to recognize the side of amicus brief.

- Manual labeling + binary classifier to automate.
- Gaussian prior.
- For ϕ_i and Δ_k: LDA over joint amicus briefs.

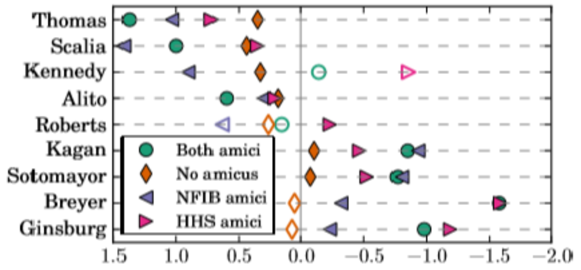
To infer:

1. Infer ϕ_i and Δ using LDA.
2. Fixe ϕ and Δ to their posterior mean and then solve for θ and other param. using an hybrid MCMC algorithm.



Decomposition of a decision per judge and factors (ideal point, amici for both sides, and combined). In this specific case, the briefs shift the ideal point toward Maple side and for three Justices, enough to change the initial side.

The scale represent a log-odd of vote.



Counterfactual analysis answering the question “What whould have been the probabilities of vote if one or both amicus were not filled?”.

The scale represent a log-odd of vote.

Extension to the Amicus Effect

An extension was proposed by [IHKR16]. Consider that the court opinions can be expressed in the same space as the Justices.

Use NLP to model Ideal Point as topic mixture over the opinion juges wrote.

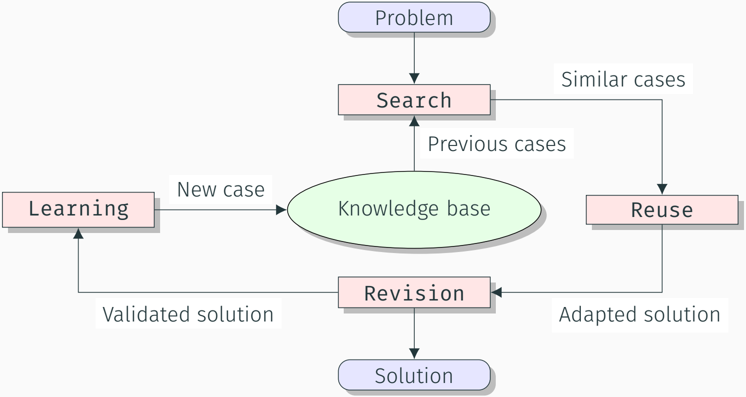


Figure 2: Illustration of the cycle carried out by a Case-based system to solve a problem as illustrated by [AP94]

Case-Based Reasoning

Case representation:

- too abstract $\implies$ poor analogy
- too concrete $\implies$ anecdotical evidence

Assumption: Legal Practitioners reason by analogy

- Still discuss among experts... [Wei05, Pos05, Kay05, Bec73]
- ... but confirmed by some studies

More efficient and reliable than Rule-base in legal domain [Kow91]

CATO [AA97, Ash88, Ash02]

- Ancester of Abstract Argumentation.
- Purpose: to teach students argumentation.
- Limite to Trade Secret Law.
- Database of textual summary and factors.
- Static factor hierarchy.

CATO : Basic reasoning moves

- Analogizing a problem to a past case with favorable outcome
- Analogizing a problem to a past with with an unfavorable outcome
- Downplaying the significance of a distinction
- Emphasizing the significance of a distinction
- Citing a favorable case to highlight strenghts
- Citing a favorage case to argue that weaknesses are not fatal
- Citing a more on point counterexample to a case city by an oponent
- Citing an as on point counterexample.

CATO : process of justification

1. Process to justify a favorable decision on an issue
2. Point to strengths related to an issue and why it matters
3. Show favorable cases
4. Discuss weaknesses and compensating factors
5. Show cases with favorable outcome but with the same weaknesses

What conclusions to draw ?

- Attitudinalism validated by many studies
- There is a room of improvement to correct bias
- To many restriction in Expert Systems
- Need for NLP and flexible approach

PLAN

Legal Analysis

Abstract Argumentation

- Dung's Abstract Argumentation
- Extensions & Generalizations
- Weighted Argumentation Framework
- Applications to Legal Domain

Sequential Decision Processes

Perspectives

According to [CLS05]:

1. Building the arguments, i.e. defining the arguments and the relation(s) between them
2. Valuating the arguments using their relations, a strength, etc.
3. Selecting some arguments using a criterion (a *semantic*) depending on the problem we want to solve or the situation to model.

Definition (Abstract Argumentation Framework [Dun95])

An AAF is a pair $F = (A, R)$ where:

1. A is a non-empty set of arguments.
2. $R \subseteq A \times A$, i.e. a binary relation on A.

Let $(a, b) \in A^2$, $a\, R\, b$ indicates that a *attacks* b.

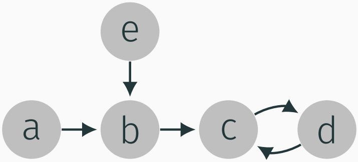


Figure 3: A graph representation of an AAF.

Definition (Attack to and from a set)

Given an AAF (A, R), $a \in A$, $S \subseteq A$, then:

1. S attacks a iff $\exists b \in S$ such that $b\,R\,a$.
2. a attacks S iff $\exists b \in S$ such that $a\,R\,b$.

semantic: how to solve the conflicts between arguments.

Definition (Conflict-free Set)

Given an AAF $F = (A, R)$ and a set $S \subseteq A$, S is conflict-free in F if $\forall (a, b) \in S^2,\ (a, b) \notin R$.

Definition (Admissible Set)

Given an AAF $F = (A, R)$ and a set $S \subseteq A$, S is admissible in F if S is conflict-free and each a in S is defended by S in F.

Definition (Extension)

An extension is defined as an admissible set in F.

Notation

We denote by $\mathbf{E} = \{\varepsilon_i\}_i$ the set of all possible extensions on an AAF F.

Notation

For a given AAF F, we define the characteristic function of F as the total operator $\gamma_F : 2^A \rightarrow 2^A$, defined as $\gamma_F(S) = \{a \in A \mid a \text{ is defended by } S \text{ in } F\}$.

Definition (Complete Extension)

$\varepsilon \in \mathsf{E}$ is *complete* iff $\forall a \in \gamma_F(\varepsilon),\ a \in \varepsilon$.

Definition (Preferred Extension)

$\varepsilon \in \mathsf{E}$ is *preferred* iff ε is maximal in A (w.r.t. the set inclusion $\subseteq$), i.e. $\forall \varepsilon' \subseteq \mathsf{E},\ \varepsilon \neq \varepsilon',\ \varepsilon \not\subset \varepsilon'$.

Definition (Grounded Extension)

$\varepsilon \in \mathsf{E}$ is the unique *grounded extension* iff ε is the least fix-point for γ_F (w.r.t. the set inclusion $\subseteq$).

Definition (Stable Extension)

$\varepsilon \in \mathsf{E}$ is stable iif $\forall a \in A \setminus \varepsilon,\ \exists b \in S,\ (b, a) \in R$.

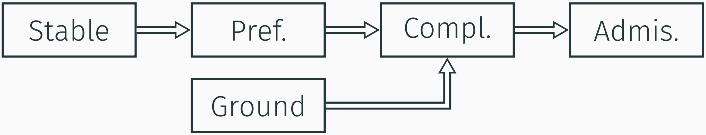
Definition (Well-Founded Argumentation Framework)

An AAF F is well-founded iff there is no infinite sequence of arguments i.e. $a = (a_i)_{i \in \mathbb{N}},\ (a_i, a_{i+1}) \in R$.

If A is finite, a well-founded AAF is represented by an acyclic graph.

Properties

If F is a Well-Founded Argumentation Framework, it has exactly one extension that is grounded, stable, prefered and complete at the same time.



Definition (Acceptability of an argument)

Let F be an AAF and $x \in A$ an argument. With regard to a semantic σ defining a set of extension E_σ:

- **Skeptical**: x is skeptically accepted iff $\forall \varepsilon \in \mathsf{E}_\sigma,\ x \in \varepsilon$, i.e. the argument belongs to all the extensions of the semantic.
- **Credulous** : x is credulous accepted iff $\exists \varepsilon \in \mathsf{E}_\sigma,\ x \in \varepsilon$, i.e. the argument is at least in one extensions.

Very large and active litterature…

Attack Frameworks

- Dung's Frameworks (AF) [Dun95]: $F = (A, R)$ with $R \subseteq A \times A$.
- Framework with Sets of Attacking Arguments (SETAF) [NP07]: $F = (A, R)$ with $R \subseteq (2^A \setminus \emptyset) \times A$.
- Framework with Recursive Attack (AFRA) [BCGG11]: $F = (A, R)$ with $R \subseteq A \times 2^{A \cup R}$.
- Extended Argumentation Framework (EAF) [MP10]: $F = (A, R, D)$ with $R \subseteq A \times A$ and $D \subseteq A \times R$.

Very large and active litterature...

Support Frameworks

- Bipolar Argumentation Framework (BAF) [CLS05]: $F = (A, R, S)$ with $R \subseteq A \times A$ and $D \subseteq A \times A$.
- Argumentation Framework with Necessities (AFN) [NR11]: $F = (A, R, N)$ with $R \subseteq A \times A$ and $D \subseteq (2^A \setminus \emptyset) \times A$.
- Evidential Argumentation System (EAS) [ON08]: $F = (A, R, E)$ with $R \subseteq (2^A \setminus \emptyset) \times A$ and $E \subseteq (2^A \setminus \emptyset) \times A$.
- Abstract Dialectical Framework (ADF) [BES+13]: $F = (A, R, C)$ with $R \subseteq A \times A$ and $C = \{C_a\}_{a \in A}$ a set of acceptance conditions.

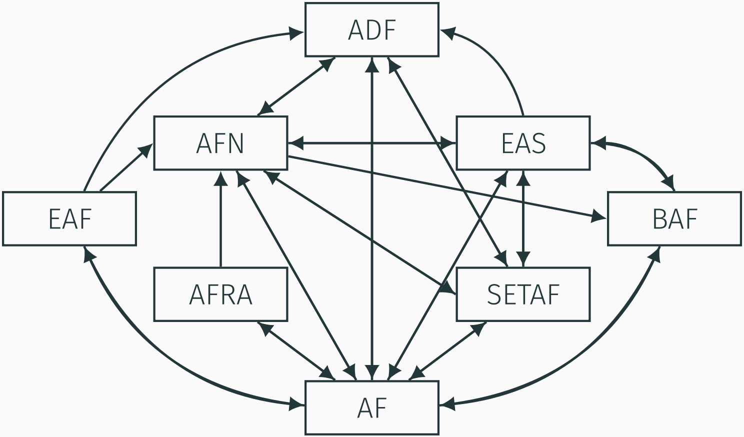


Figure 4: Relation of translatibility between Abstract Argumentation Frameworks. The relations cover different type of translation. See [Pol16].

- Weighted Argumentation Framework [DHM+11]
- Abstract Dialectical Frameworks [BES+13]
- Evidential-based Argumentation Frameworks [Ore07]

Definition (Weighted Argumentation Framework)

A WAF is a triple $F = (A, R, w)$ where w is a function such that $w : R \to \mathbb{R}^+$.

Allow the usage of an *inconsistency* budget to generalize extensions (relaxe the conflict-free def.).

More expressive than ([DHM+11]):

- Preference-Based Framework (PAF) [AC98]
- Value-Based Argumentation Framework [BC03]
- Extended Argumentation Frameworks [MP10]

Definition (Abstract Dialectical Framework (ADF))

An ADF is a tuple $F = (A, R, C)$ where:

1. A is a set of arguments.
2. $R \subseteq A \times A$.
3. $C = \{C_a\}_{a \in A}$, a set of functions such that $C_a : \mathcal{P}(\text{pred}(x)) \to \{t, f\}$.

Need another notion for extension: models!

Definition (Interpretation and models)

Given a set of elements A:

- A **three-value interpretation** v is a mapping from $\{\phi_a\}$ to $\{t, f, u\}$. The set of three-value interpretations is denoted K_3
- A **three-value model** v of A is an interpretation such that $\forall a \in A,\ v(a) \neq u \implies v(a) = v(\phi_a)$. The set of three-value models over A is denoted K_3^A

Information ordering

$\leq_i$ such that $u \leq_i t$ and $u \leq_i f$

$(\{t, f, u\}, \leq_i)$ a meet-semilattice with the "consensus" meet $\sqcap$ such that $f \sqcap f = f$ and $t \sqcap t = t$ and u otherwise.

Information ordering on K_3^A:

$\forall v_1, v_2 \in K_3^A,\ v_1 \leq_i v_2 \leftrightarrow \forall a \in A,\ v_1(a) \leq_i v_2(a).$

$(K_3^A, \leq_i)$ a meet-semilattice with the consensus meet $\sqcap$ such that $v_1 \sqcap v_2 = v_1(a) \sqcap v_2(a),\ \forall a \in A.$

Remark: The least element of $(K_3^A, \sqcap)$ is the mapping that maps to any element of A the value undecidable.

Definition (Interpretation extension)

$w \in K_2^A$ extends $v \in K_3^A$ iif $v \leq_i w$. $[v]_2$ denotes the set of two-value interpretation extending w.

Definition (Grounded Model)

Given an ADF $F = (A, C)$ and $v \in K_3^A$ the grounded extension is the least fixed point of the operator

$$\Gamma_F(v) : a \mapsto \sqcap\{w(\phi_a \mid w \in [v]_2\}$$

The fixed point exists and is generally three-valued [BES+13].

Definition (Acceptability Model)

Given an ADF $F = (A, C)$ and $v \in K_3^A$, then

- v is admissible iff $v \leq_i \Gamma_F(v)$;
- v is complete iif $\Gamma_F(v) = v$;
- v is preferred iif v is $\leq_i$-maximal admissible.

Combining Abstract Argumentation with Subjective Logic.

Definition (Opinion)

An opinion ω about a proposal ϕ is a triple $\omega(\phi) = (b(\phi), d(\phi), u(\phi))$ where $b(\phi)$ (resp. $d(\phi)$, $u(\phi)$) is the level of belief that ϕ holds (resp. disbelief, unecrtainty), such that $b(\phi) + d(\phi) + u(\phi) = 1$ and $b(\phi), d(\phi), u(\phi) \in [0, 1]$.

Definition (Opinion Operators)

- Negation: $\neg\omega(\phi) = (d(\phi), b(\phi), u(\phi))$.
- Recommendation:
  $\omega(\phi) \otimes \omega(\psi) = (b(\phi)b(\psi), b(\phi)d(\psi), d(\phi) + u(\phi) + b(\phi)u(\psi))$.
- Consensus: $\omega_A(\phi) \oplus \omega_B(\phi) = \left(\frac{b_A(\phi)u_B(\phi)+u_A(\phi)b_B(\phi)}{k}, \frac{d_A(\phi)u_B(\phi)+u_A(\phi)d_B(\phi)}{k}, \frac{u_A(\phi)u_B(\phi)}{k}\right)$ with $k = u_A(\phi) + u_B(\phi) - u_A(\phi)u_B(\phi)$

- Quantitative Argumentation Debate (QuAD) [BRT$^+$15]
- Discontinuity-Free QuAD (DF-QuAD) [RTAB16]
- Social Abstract Argumentation [LM11]
- Compensation-based semantics [ABDV16]

- Probabilistic Jury-based Dispute Resolution [DT10]
- Abstract Argumentation for Case-Based Reasoning [vST16, ASL+15, OnP08]

Definition (Case, Case Base, New Case)

Given a set of features $\mathbb{F}$, possibility infinite, and a binary case outcome $O = \{+, -\}$

- a Case is a pair (X, o) with $X \subseteq \mathbb{F}$ and $o \in O$,
- a Case Base is a finite set $\text{CB} \subseteq \mathcal{P}(\mathbb{F}) \times O$ of cases such that for $(X, o_X), (Y, o_Y) \in \text{CB}$ if $X = Y$, $o_X = o_Y$,
- a New Case is a set $N \subseteq \mathbb{F}$.

Definition (Nearest Cases)

Given a CB and a new case N, a past case $(X, o_X) \in \text{CB}$ is nearest to N if X is maximal for the $\subseteq$-inclusion.

Definition (AF associated to a Case-Base)

Given a CB, a default outcome d and a new case N, the associated Argumentation Framework $F_{\mathrm{CB}} = (A, R)$ is built such that

- $A = \mathrm{CB} \cup \{(N, ?)\} \cup \{(\emptyset, d\}$,
- for $(X, o_X), (Y, o_Y) \in \mathrm{CB} \cup \{(\emptyset, d\}$ it holds that $((X, o_X), (Y, o_Y)) \in R$ iif:
  1. $o_X \neq o_Y$ (different outcome)
  2. $Y \not\subseteq X$ (specificity)
  3. $\nexists (Z, o_X) \in \mathrm{CB}$ with $Y \not\subseteq Z \not\subseteq X$ (concision)
- for $(Y, o_Y) \in \mathrm{CB}$, $((N, ?), (Y, o_Y)) \in R$ holds iif $y \not\subseteq N$

Definition (AA outcome)

The AA outcome of a new case N is $d \times \mathbb{1}_{((\emptyset,d)\in \text{ground}(F_{\text{CB}}))} + \bar{d}(1 - \mathbb{1}_{((\emptyset,d)\in \text{ground}(F_{\text{CB}}))})$

Another approach by learning rules: [ASL+15]

Example

From $C_1 = (\{\}, -)$ (default case) and $C_2 = (\{F_1\}, -) \implies F_1$ is not relevant to the judges.
Third case $C_3 = (\{F_1, F_2\}, +)$, as it is reversed between C_2 and C3, the conjunction of F_1 and F_2 is important. If we had a case $C_4 = (\{F_2\}, +)$, we can deduce that F_1 is irrelevant and the conjunction is not important, F_2 is enough

Multi-agent approach [OnP08]:

Definition (Multi-agent Case Base Reasoning Systems)

A Multi-agent Case Base Reasoning Systems is a tuple $M = ((A_1, C_1), ..., (A_n, C_n))$ where A_i is an agent with a case base $C_i = \{c_i, ..., c_{m_i}\}$. A previously, a case c is a tuple (X, o_x) with $X \subseteq \mathbb{F}$ and $o_x \in S = \{S_1, ..., S_k\}$ the outcome among k classes.

Definition (Justified Prediction)

A Justified Prediction is a tuple $J = (A, N, s, D)$ where agent A consider s the correct class for a new case case N because of the $N \subseteq D$, i.e D is more general than N.

PLAN

Legal Analysis

Abstract Argumentation

Sequential Decision Processes

- Markov Decision Process Models
- Decentralized Control
- Non-Stationary Environments

Perspectives

Definition (Markov Decision Process (intrinsic form))

A Markov Decision Process (MDP) is a tuple (S, A, T, p, r) where

- S is the (finite and discrete) state space,
- A is the (finite and discrete) set of actions ,
- T defining the space of time with $0, ..., T$,
- p a probability measure over S given $S \times A$, i.e. $p(s, a, s') = \mathbb{P}(s \mid a, s')$,
- r a reward function defined by $r : S \times A \to \mathbb{R}$

with p holding the (weak) Markov property, i.e.
$\forall h_t = (s_0, a_0, ..., s_{t-1}, a_{t-1}, s_t)$,
$\mathbb{P}(s_{t+1} \mid a_t, h_t) = \mathbb{P}(s_{t+1} \mid a_t, s_t) = p(s_{t+1}, a_t, s_t)$

Control policy: $g_t : S^t \times A^{t-1} \to A$

Definition (MDP (dynamical form))

A Markov Decision Process (MDP) dynamic model is defined by:

- System dynamic: $X_{t+1} = f_t(X_t, U_t)$,
- Control process: $U_t = g_t(X_{1:t}, U_{1:t-1})$,

and consists in finding $g^* = \arg\min_{g} R(g)$

with e.g. γ-ponderate criterion: $R(g) = \mathbb{E}^g[\sum_{t=0}^{T} \gamma^t r_t^g],\ \gamma \in]0,1[$

Bellman's property

MDP optimal policies are markovian policies: $g_t : S \to A$

The Bellman equation is given by:

$$\forall s \in S,\ g^*(s) = \arg\min_a \{r(s,a) + \gamma \sum_{s' \in S} p(s,a,s') V^{g^*}(s')\}$$

with $V^{g_t}(s) = r_t^g + \gamma \sum_{s' \in S} p(s, g_t(s), s') V^{g_{t+1}}(s')$ the value function of a policy.

Definition (Partially Observable Markov Decision Process)

A POMDP is a tuple (S, A, O, T, p, q, r) where

- S is the (finite and discrete) state space,
- A is the (finite and discrete) set of actions,
- O is the (finite and discrete) set of observations,
- T defining the space of time with $0, ..., T$,
- p a probability measure over S given $S \times A$, i.e. $p(s, a, s') = \mathbb{P}(s \mid a, s')$,
- q a probability measure over O given $S \times A$, i.e. $q(o, a, s) = \mathbb{P}(o \mid a, s)$,
- r a reward function defined by $r : S \times A \to \mathbb{R}$

with p holding the (weak) Markov property.

Same results.

In practice, there are many ways to solve such a dynamic program: linear programming, value-iteration, policy-iterations. See in particular [SB08, Put94]

OTHER MODELS OR EXTENSIONS

- Mixed Observability MDP
- Possibislist MDP
- Algebraic MDP

Less litterature, less optimality results, but seems promising to be coupled with Abstract Argumentation and non-monotonic reasoning.

Much harder than centralized [Rad62, **?**]:

- POMDP formalism
- *n* controlers instead of 1
- Very simple counter example: [Wit73]
- No general optimality results until 2013 [NMT10, NMT13, MNT08]

DECENTRALIZED CONTROL

Some characteristics:

- Uncertainty (environment and controller)
- Information asymmetry
- Signaling
- Information growth

Many studies for particular information type:

- delayed sharing information structure [Wit71],
- delayed state sharing [NMT10, ADM87],
- partially nested systems [HC72],
- periodic sharing information structure [OVLW97],
- belief sharing information structure [Yuk09],
- finite state memory controllers [ABZ12],
- broadcast information structure [WL10]

Formalism:

- n controllers
- $\{X_t\}_{t=0}^{\infty},\ X_t \in \mathcal{X}$ state process
- $\forall i,\ i \in \{1, ..., n\},\ \{Y_t^i\}_{t=0}^{\infty},\ Y_t^i \in \mathcal{Y}^i$ observation process
- $\{U_t^i\}_{t=0}^{\infty},\ U_t^i \in \mathcal{U}^i$ control process
- $\{R_t\}_{t=0}^{\infty}$ reward process

- X is a controled markov process
- R_t depends on X_t, X_{t+1}, U_t
- Y_t depends on X_t, U_{t-1}

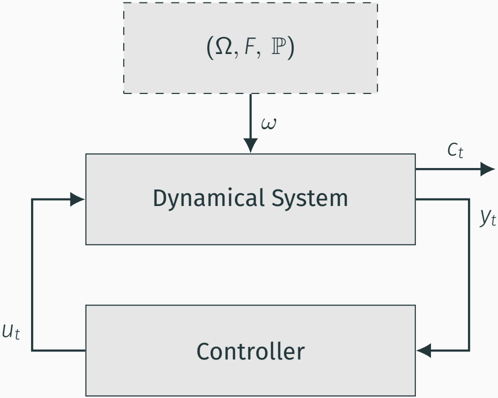


Figure 5: Dynamical Model

Information structure

$$\{Y_t^i, U_t^i\} \subseteq I_t^i \subseteq \{\mathsf{Y}_t, \mathsf{U}_t\}$$

matrix of controllers information: $(\mathsf{I}_t^i)_{1 \leq i \leq n; t \geq 0}$

Control strategy

$$g_t^i : I_t^i \to U_t^i$$

Decentralized Control problem:

$$\mathbf{g}^* = \arg\max_{\mathbf{g}} \mathbb{E}^{\mathbf{g}} [\sum_{t=0}^{\infty} \beta^t R_t] \quad (1)$$

with $\beta \in [0, 1]$.

Centralized is a special case of decentralized problems:

- if $n = 1 \implies$ POMDP
- if 1. + $Y = X \implies$ MDP

How to solve the general case ?

- Person-by-person approach
- Common information approach

Common information approach

$C_t = \bigcap_{\tau \leq t} \bigcap_{i=1}^{p} I_\tau^i$

- local information: $L_t^i = I_t^i \setminus C_t$
- $U_t^i = C_t \cup L_t^i,\ \forall i \in [1, n]$
- $C_t \subseteq C_{t+1}$

"Local" policy $\gamma_t^i = L_t^i \mapsto U_t^i$ (prescription).

Definition (Partial History Sharing)

An informations structure is a partial history sharing structure iif:

- For any set of realization A of L^i_{t+1} and any realization c_t, l^i_t, u^i_t, y^i_{t+1} of, respectively, C_t, L^i_t, U^i_t, U^i_{t+1} and Y^i_{t+1}:

$$\begin{aligned} & \mathbb{P}(L^i_{t+1} \in A \mid C_t = c_t, L^i_t = l^i_t, U^i_t = u^i_t, Y^i_{t+1} = y^i_{t+1}) \\ = \; & \mathbb{P}(L^i_{t+1} \in A \mid L^i_t = l^i_t, U^i_t = u^i_t, Y^i_{t+1} = y^i_{t+1}) \end{aligned}$$

- The space of realization of L^i_t, denoted $\mathcal{L}^i_t$, is uniformly bounded:

$$\exists k \in \mathbb{N}, \; \forall i \in [1, n], \; |\mathcal{L}^i_t| \leq k \tag{2}$$

Resolutions steps:

- Construct an equivalent *coordinated* system:
  - At time t it choses prescriptions: $\Gamma_t = \{\Gamma_t^i\}_{1 \leq i \leq n}$ such that $U_t^i = \Gamma_t^i(L_t^t)$
  - Coordination law: $\Phi_t : C_t \to (\Gamma_t^i)_{1 \leq i \leq n}$
  - Control strategy: $\mathbf{g}_t^i = \{g_t^i\}_{t>0}, \ \forall i \in [0, n]$ with $g_t^i(c_t, l^i) = \Phi_t^i(c_t)(l^i)$

  as $R(\Phi) = R(g)$, finding $g^* \Leftrightarrow \Phi^*$
- Identify an information state.

Resolutions steps:

- Construct an equivalent *coordinated* system.
- Identify an information state: enough to look for $\Phi_t : Z_t \mapsto (\Gamma_t^i)_{0 \leq i \leq n}$ with $\{Z_t\}_{t=0}^{\infty}$ an information state.

$$\Phi^*(z) = \arg\sup_{\gamma} Q(z, \gamma), \ \forall z \in \mathcal{Z} \tag{3}$$

where Q is the unique fixe-point to the following system:

$$Q(z, \gamma) = \mathbb{E}[R_t + \beta V(Z_{t+1}) | Z_t = z, \Gamma_t^1 = \gamma_t^1, ..., \Gamma_t^n = \gamma_t^n], \ \forall z \in \mathcal{Z}, \ \forall \gamma$$

$$V(z) = \sup_{\gamma} Q(z, \gamma), \ \forall z \in \mathcal{Z}$$

Limitation of POMDP formalism: stationarity of X, $\mathcal{X}$, R, etc.

$\implies$ not suitable for Justice (jurisprudence, disruption, etc.)

Definition (Hidden-Mode Markov Decision Process [CYZ99])

A HM-MDP is a tuple (M, C) where

- $M = \{m_1, ..., m_N\}$ a set of modes with $m_i = (S, A_i, p_i, r_i)$ is a MDP,
- C is a probably measure over M.

A mode = stationary environment.

Definition (Hidden Semi-Markov-Mode Markov Decision Processes [HBW14])

A (HS3MDP) is a tuple (M, C, H) where

- (M, C) is an HM-MPD,
- H is a probably measure over $\mathbb{N}$ given two modes, i.e. $H(m, m', n)$ is the probability to stay n timesteps into m' coming from m.

Mode transition, given an initial mode m and mode duration k:

1. Stay k timesteps in m.
2. Draw a new m according to C. Draw a new k according to H.
3. Repeat from 1.

- $N = 1$, HM-MDP $\Leftrightarrow$ MDP
- $N > 1$, HM-MDP $\Leftrightarrow$ POMDP
- $\forall N$, HM-MDP $\Leftrightarrow$ HS3MDP

Several questions:

- How to learn the environment ?
- How to detect the mode changes ?

Reinforcement Learning with Context Detection algorithm

Change Point Detection

- Sequential Analysis: assume known processes
- Time-serie Clustering: assume known number of processes [KRMP16, KR13, KR12]

Sequential Analysis: CUSUM [BN93]

X generated by μ_1 then μ_2. At time t, $(x_0, x_1, ..., x_t, x_t)$

"H_0: the distribution is μ_1"

$$S_t = \max(0, S_{t-1} + \ln(\frac{\mu_2(x_t)}{\mu_1(x_t)}))$$

with $S_0 = 0$.

$S_t > \delta$, reject H_0

$c = \ln \frac{1-\beta}{\alpha}$ [Wal45].

Argumentation problems with Probabilistic Strategies [HBM+15, Hun14]

The main conclusion is: “information is what matters the most”

- In economic models $\implies$ (omniscient hypothesis $\leftrightarrow$ solving by construction the problems) [VH37, Hay45]
- In Abstract Argumentation $\implies$ create the concrete arguments, changes in strategies, CBR.
- In Control theory $\implies$ different optimality results.
- In Justice system $\implies$ influence of amicus, judges ideology.

PLAN

Legal Analysis

Abstract Argumentation

Sequential Decision Processes

Perspectives

- Room of improvement
- Simulation Based Reasoning (SBR)

- forecast justified by legal terms rather than binary outcome
- explanation generation
- automatic NLP data gathering and processing

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